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THE ELASTIC FIELD GENERATED BY TWO LOADS MOVING ALONG TWO STRINGS ON AN ELASTICALLY SUPPORTED MEMBRANE

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The uniform motion of two constant loads moving in opposite directions along two infinite parallel strings on an elastically supported membrane is studied. The problem is analyzed for arbitrary ratios of the velocities of the loads and the wave velocities in the strings and the membrane. The displacements of the system are calculated and presented as deflection profiles of the strings and membrane. Further, the lateral forces are determined, acting at the strings and loads due to the interaction through the membrane. For subcritical velocities of the loads, an analogy is shown with gravitational interaction. For cases in which one of the loads moves transcritically or supercritically, the lateral force acting at the other load is shown to have an impact character.

1. INTRODUCTION

For the design of two parallel high-speed tracks it is of practical interest to understand the peculiarities of the wave-processes in the sub-soil and tracks when two high-speed trains are passing each other at sub-, trans- or supercritical velocities. In particular, the influence of one of the tracks on the dynamic response of the other is relevant. In fact, when two trains are passing at a subcritical speed the stationary spatial eigenfields, which are moving with the trains, partly overlap especially when the distance between the tracks is small. Further, the resulting interaction force has a lateral component, which might affect the stability of continuous welded track. Moreover, when the train is moving trans- or supercritically it generates waves in the sub-soil and (or) the track due to the Mach-effect. In this case, the excited wave field is not localized near the train, but propagates along the surface. It is clear that this wave field can affect the other track and train and furthermore, due to the reflections from the other track, affect the source. The main purpose of this paper is to demonstrate qualitatively these phenomena.

The analysis of the wave process in a track on a sub-soil due to a high-speed train has recently been investigated by using a model consisting of a uniformly moving load along a Euler–Bernoulli beam model on a half-space [1–3]. The importance of the interaction of the beam-waves and the surface waves was shown. It is too complicated to extend this model for the analysis of the case representing a double track. However, for qualitative

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understanding of the processes in both the track and the surface it is sufficient to consider a simpler model of the ‘‘track–subsoil’’ interaction. Therefore, the model is extended to consist of a string on an elastically supported membrane, which was introduced in reference [4].

In this paper, the model is extended to two parallel strings on an elastically supported membrane. This model can be used to study the mechanical interaction through the surface between two objects moving along the string.

Here the steady–state behaviour of the system due to two constant loads moving in opposite directions along the strings will be determined. The lateral forces on the strings and loads will be determined for sub-, trans- and supercritical velocities, showing a kind of analogy, for the subcritical velocities, with Newton's law of gravitational interaction. For trans- and supercritical motions, the lateral forces show an impulsive character.

2. MODEL

Two identical infinite parallel strings on an elastically supported membrane are considered. Along each string a constant load is uniformly moving in an opposite direction as depicted in Figure 1, where d' is the distance between the strings. It is assumed that the strings are in continuous contact with the membrane at the lines $y' = -d/2$, $y' = d'/2$. The tension in the density of each string is the same.

The equations for the vertical displacements of the coupled system of the elastically supported membrane and the strings are

$$U'^m_{t't'} - c_m^2\,(U'^m_{x'x'} + U'^m_{y'y'}) + \mu^2 U' = -(1/\rho')\{\delta(y' - d'/2)\{U'^{s1}_{t't'} - c_s^2\, U'^{s1}_{x'x'} + \tilde{P}_1\,\delta(x' + V_1\,t')\}$$
$$+\,\delta(y' + d'/2)\{U'^{s2}_{t't'} - c_s^2\, U'^{s2}_{x'x'} + \tilde{P}_2\,\delta(x' - V_2\,t')\}\},$$
$$-\infty < x', y', t' < +\infty,$$
$$U'^m(x', y' = d'/2, t') = U'^{s1}(x', t'), \qquad U'^m(x', y' = -d'/2, t') = U'^{s2}(x', t'). \tag{1}$$

Here $c_m^2 = N^m/\rho^m$, $c_s^2 = N^s/\rho^{s2}$, $\mu = k/\rho^m$, $\rho' = \rho^m/\rho^s$, $\tilde{P}_1 = P_1'/\rho^s$, $\tilde{P}_2\,\tilde{P}_2'/\rho^s$; $U'^m(x, y, t)$ and $U'^{s1}(x, t)$, $U'^{s2}(x, t)$ are the vertical displacements of the membrane and the strings; respectively, N^m, N^s are the membrane and the string tensions, respectively; c_m, c_s are the wave velocities in the membrane and in the string, respectively; k is the stiffness of the elastic foundation of the membrane per unit square; ρ^m is the mass of the membrane per

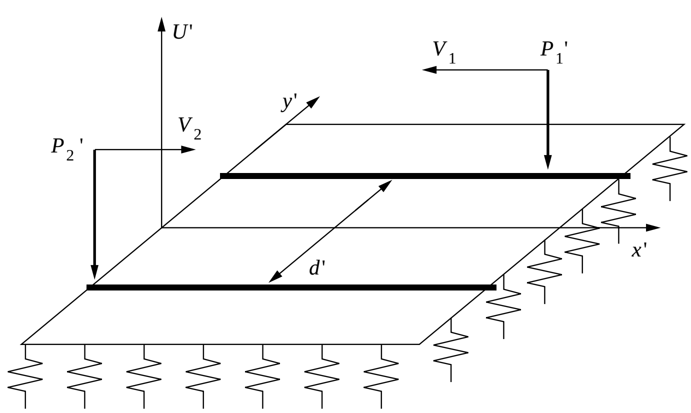

Figure 1. Model and reference system.

unit area and ρ^s is the mass of the string per unit length; and P_1', P_2' are the constant loads. The equations (1) are put into dimensionless form by introducing

$$t = t'\mu, \qquad x, y = x'\mu/c_s, y'\mu/c_s, \qquad d = d'\mu/c_s, \qquad U^{m,sj} = U'^{m,sj}\mu/c_s, \qquad j = 1, 2,$$

resulting in

$$U^m_{tt} - \alpha^2(U^m_{xx} + U^m_{yy}) + U = -(1/\rho)\{\delta(y - d/2)\{U^{s1}_{tt} - U^{s1}_{xx} + P_1\,\delta(x + \beta_1 t)\}$$
$$+ \delta(y + d/2)\{U^{s2}_{tt} - U^{s2}_{xx} + P_2\,\delta(x - \beta_2 t)\}\},$$
$$-\infty < x, y, t < +\infty,$$
$$U^m(x, y = d/2, t) = U^{s1}(x, t), \qquad U^m(x, y = -d/2, t) = U^{s2}(x, t), \tag{2}$$

where $\alpha = c_m/c_s$, $\quad \rho = (c_s/\mu)\rho'$, $\quad \beta_j = V_j/c_s$, $\quad P_j = \tilde{P}_j/c_s^2$, $\quad j = 1, 2$.

3. GENERAL SOLUTION

To solve the system (2) it is sufficient to derive the solution for one load only, for instance for the load P_1 ($P_2 = 0$). Then the solution for two loads can be obtained by superposition. We therefore apply the following exponential Fourier transforms over time and spatial co-ordinates to system (2) with $P_2 = 0$. (U^m_1 denotes the displacement of the membrane):

$$W^m(\omega, k_1, k_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U^m_1(x, y, t)\exp\{i(\omega t - k_1 x - k_2 y)\}\,dt\,dx\,dy,$$

$$W^{sj}(\omega, k_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U^{sj}(x, t)\exp\{i(\omega t - k_1 x)\}\,dt\,dx, \qquad j = 1, 2.$$

This gives

$$W^m(\omega, k_1, k_2) = -\frac{1}{\rho D_m(\omega, k_1, k_2)}\left\{D_s(\omega, k_1)\exp\left(-i\frac{k_2 d}{2}\right)W^{s1}(\omega, k_1)\right.$$
$$\left. -2\pi P_1\,\delta(\omega + k_1\beta_1)\exp\left(-i\frac{k_2 d}{2}\right) + D_s(\omega, k_1)\exp\left(i\frac{k_2 d}{2}\right)W^{s2}(\omega, k_1)\right\}, \tag{3}$$

where $D_m(\omega, k_1, k_2) = \omega^2 - \alpha^2(k_1^2 + k_2^2) - 1$ is the dispersion relation of the membrane on the elastic foundation and $D_s(\omega, k_1) = \omega^2 - k_1^2$ the dispersion relation of the string. The relation between the image of the membrane displacements and the images of the string displacements, representing the compatibility conditions in system (2), gives

$$\int_{-\infty}^{\infty} W^m(\omega, k_1, k_2)\exp\left(+i\frac{k_2 d}{2}\right)dk_2 = 2\pi W^{s1}(\omega, k_1),$$

$$\int_{-\infty}^{\infty} W^m(\omega, k_1, k_2)\exp\left(-i\frac{k_2 d}{2}\right)dk_2 = 2\pi W^{s2}(\omega, k_1).$$

Substituting the expression for $W^m(\omega, k_1, k_2)$ into the compatibility conditions, one obtains a linear system of algebraic equations with respect to W^{s1} and W^{s2}:

$$W^{s1}(\chi^{-1}(\omega, k_1) + D_s(\omega, k_1)) + W^{s2} D_s(\omega, k_1)\exp\left(\mathrm{i}\frac{d}{\alpha}\sigma(\omega, k_1)\right) = 2\pi P_1\,\delta(\omega + k_1\beta_1),$$

$$W^{s1} D_s(\omega, k_1)\exp\left(\mathrm{i}\frac{d}{\alpha}\sigma(\omega, k_1)\right) + W^{s2}(\chi^{-1}(\omega, k_1) + D_s(\omega, k_1))$$

$$= 2\pi P_1\,\delta(\omega + k_1\beta_1)\exp\left(\mathrm{i}\frac{d}{\alpha}\sigma(\omega, k_1)\right), \tag{4}$$

where

$$\chi(v, k_1) = \frac{1}{2\pi\rho}\int_{-\infty}^{\infty}\frac{\mathrm{d}k_2}{D_m(\omega, k_1, k_2)} = \frac{1}{\mathrm{i}\gamma\sigma(\omega, k_1)},$$

$$\gamma = 2\alpha\rho, \qquad \sigma(\omega, k_1) = \sqrt{\omega^2 - \alpha^2 k_1^2 - 1}, \qquad \mathrm{Im}\sqrt{\omega^2 - \alpha^2 k_1^2 - 1} > 0.$$

Solving the system of equations (4), one finds for W^{s1} and W^{s2},

$$W^{s1} = \frac{2\pi P_1\,\delta(\omega + k_1\beta_1)}{S^2(\omega, k_1) - \left(D_s(\omega, k_1)\exp\left(\mathrm{i}\frac{d}{\alpha}\sigma(\omega, k_1)\right)\right)^2}$$

$$\times\left\{S(\omega, k_1) - D_s(\omega, k_1)\exp\left(\mathrm{i}\frac{2d}{\alpha}\sigma(\omega, k_1)\right)\right\},$$

$$W^{s2} = \frac{2\pi i\gamma\sigma(\omega, k_1)P_1\,\delta(\omega + k_1\beta_1)}{S^2(\omega, k_1) - \left(D_s(\omega, k_1)\exp\left(\mathrm{i}\frac{d}{\alpha}\sigma(\omega, k_1)\right)\right)^2}\exp\left(\mathrm{i}\frac{d}{\alpha}\sigma(\omega, k_1)\right), \tag{5}$$

with $S(\omega, k_1) = D_s(\omega, k_1) + \mathrm{i}\gamma\sigma(\omega, k_1) = \omega^2 - k_1^2 + \mathrm{i}\gamma\sqrt{\omega^2 - \alpha^2 k_1^2 - 1}$ being the dispersion relation of one string interacting with the elastically supported membrane. Therefore, the determinant of system (4),

$$\det(\omega, k_1) = (\omega^2 - k_1^2 + \mathrm{i}\gamma\sqrt{\omega^2 - \alpha^2 k_1^2 - 1})^2 - (\omega^2 - k_1^2)^2\exp\left(\mathrm{i}\frac{2d}{\alpha}\sqrt{\omega^2 - \alpha^2 k_1^2 - 1}\right) \tag{6}$$

is the dispersion relation of the system, two strings interacting with the elastically supported membrane. Expression (6) consists of two members. The first member describes the "one string part" [4] and the second member (with the typical exponential multiplier) describes the influence from the other string.

Substitution of the expressions for $W^{s1}(\omega, k_1)$ and $W^{s2}(\omega, k_1)$ from equations (5) into equation (3) results in

$$W^m(\omega, k_1, k_2) = \frac{2\pi i\gamma\sigma(\omega, k_1)P_1\, \delta(\omega + k_1\beta_1)}{\rho D_m(\omega, k_1, k_2)\left(S^2(\omega, k_1) - D_s^2(\omega, k_1)\exp\left(i\frac{2d}{\alpha}\sigma(\omega, k_1)\right)\right)} \times\left\{S(\omega, k_1)\exp\left(-i\frac{k_2 d}{2}\right) - D_s(\omega, k_1)\exp\left(i\frac{k_2 d}{2}\right)\exp\left(i\frac{d}{\alpha}\sigma(\omega, k_1)\right)\right\}. \tag{7}$$

Hence, the steady state solution of the membrane displacements due to a uniformly moving constant load along string 1 has the form

$$U_1^m(x, y, t) = \frac{P}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{i\gamma\sigma(k_1\beta_1, k_1)\exp\{i(x+\beta_1 t)k_1\}}{\rho D_m(k_1\beta_1, k_1, k_2)\left(S^2(k_1\beta_1, k_1) - D_s^2(k_1\beta_1, k_1)\exp\left(i\frac{2d}{\alpha}\sigma(k_1\beta_1, k_1)\right)\right)}$$
$$\times\left\{S(k_1\beta_1, k_1)\exp\left\{i\left(y-\frac{d}{2}\right)k_2\right\} - D_s(k_1\beta_1, k_1)\exp\left\{i\left(y+\frac{d}{2}\right)k_2\right\}\right.$$
$$\left.\times\exp\left(i\frac{d}{\alpha}\sigma(k_1\beta_1, k)\right)\right\}\mathrm{d}k_2\,\mathrm{d}k_1 = \frac{P_1}{2\pi}\int_{-\infty}^{\infty}$$
$$\times\frac{\exp\{i(x+\beta_1 t)k_1\}}{\left(\left((\beta_1^2-1)k_1^2 + i\gamma\sqrt{(\beta_1^2-\alpha^2)k_1^2-1}\right)^2 - (\beta_1^2-1)^2k_1^4\exp\left(i\frac{2d}{\alpha}\sqrt{(\beta_1^2-\alpha^2)k_1^2-1}\right)\right)}$$
$$\times\left\{((\beta_1^2-1)k_1^2 + i\gamma\sqrt{(\beta_1^2-\alpha^2)k_1^2-1})\exp\left\{i\left|y-\frac{d}{2}\right|\frac{\sqrt{(\beta_1^2-\alpha^2)k_1^2-1}}{\alpha}\right\}\right.$$
$$\left.-(\beta_1^2-1)k_1^2\exp\left\{i\left|y+\frac{d}{2}\right|\frac{\sqrt{(\beta_1^2-\alpha^2)k_1^2-1}}{\alpha}\right\}\exp\left(i\frac{d}{\alpha}\sqrt{(\beta_1^2-\alpha^2)k_1^2-1}\right)\right\}\mathrm{d}k_1, \tag{8}$$

with $\mathrm{Im}\sqrt{(\beta_1^2-\alpha^2)k_1^2-1} > 0$. By using this expression one can easily obtain the solution of the problem (2) for two moving loads by superposition,

$$U^m(x, y, t) = U_1^m(x, y, t) + U_2^m(x, y, t), \tag{9}$$

where the expression for $U_2^m(x, y, t)$ is given by equation (A1) in the Appendix.

From the expression for the displacement of the system of the membrame and the two strings (8), one can derive the critical velocity of the load. Integral (8) diverges for $\{x + \beta_1 t = 0, y = d/2\}$ when $\beta_1 = 1$ and $\alpha > 1, \Leftrightarrow (V_1 = c_s, c_s < c_m)$. Then the displacement of the elastic system under the load is infinite. In this case the integral (8) can be rewritten in the form

$$U^m(\xi = x - \beta_1 t, y)$$

$$= -\frac{\tilde{P}}{2\pi\sqrt{(\alpha^2 - \beta_1^2)}}\int_{-\infty}^{\infty} \exp\{iar \sinh(\chi + i\vartheta)\}\, d\chi + \{\text{non-diverging part}\}, \quad (10)$$

where

$$a = \frac{1}{\alpha}, \qquad r^2 = \frac{\xi^2}{(1 - (\beta_1^2/\alpha^2))} + y^2, \qquad \sin(\vartheta) = \frac{y}{r}, \qquad \cos(\vartheta) = \frac{\xi}{\sqrt{1 - \beta_1^2/\alpha^2}}, \qquad \beta_1 = 1.$$

Evidently the integral (10) diverges when $r = 0 (\xi = 0, y = 0)$. Moreover, if $\alpha = 1$, $\beta_1 = 1 \Leftrightarrow (V = c_s, c_s = c_m)$, the displacement of the whole system is infinite (for $t \to \infty$). It is clear, that the same holds for a one-string system [4].

For the integration of the expression (8) or (A1) one has to know the location of the singular points of the integrand, and thus has to solve the transcendental equation

$$R(k_1) = (ak_1^2 + i\gamma\sqrt{bk_1^2 - 1})^2 - a^2k_1^4 \exp(ic\sqrt{bk_1^2 - 1}) = 0, \quad \text{with } \operatorname{Im}\sqrt{bk_1^2 - 1} > 0.$$

where

$$a = \beta_j^2 - 1, b = \beta_j^2 - \alpha^2, c = 2d/\alpha, \beta_j = V_j/c_s, \quad j = 1, 2. \quad (11)$$

A convenient method to determine the number of roots of equation (11) is by use of Rouche's theorem; see reference [5]. First, one represents the function $R(k_1)$ in the complex k_1-plane as a single-valued function by introducing branch cuts along

$$-|w(k_1)| \leqslant -\eta, \eta \leqslant |w(k_1)| \quad \text{with Im } w(k_1) = 0 \quad \text{for } b > 0,$$

$$-i|w(k_1)| \leqslant -i\eta, i\eta \leqslant i|w(k_1)| \text{ with Im } w(k_1) = 0 \quad \text{for } b < 0,$$

$$\eta = 1/\sqrt{|b|}, \qquad w(k_1) = \sqrt{bk_1^2 - 1}.$$

Next one separates $R(k_1)$ in two parts as follows:

$$R(k_1) = f_+(k_1) f_-(k_1), \qquad R(k_1) = 0 \Rightarrow, \qquad f_+(k_1) = 0 \text{ and } f_-(k_1) = 0,$$

where

$$f_\pm(k_1) = g(k_1) \pm r(k_1), \qquad g(k_1) = ak_1^2 + i\gamma\sqrt{bk_1^2 - 1}, \qquad r(k_1) = ak_1^2 \exp(i(c/2)\sqrt{bk_1^2 - 1}).$$

Then the number of zeros of $R(k_1)$ is given by the sum of the zeros of $f_+(k_1)$ and $f_-(k_1)$. Consider now the contour in the complex k_1-plane as shown in Figure 2. Note that one may have to choose the radii of the semicircles quite large to have the zeros of $f_\pm(k_1)$ inside C, since the functions $g(k_1)$ and $r(k_1)$ are analytical and continuous inside the contour C and satisfy the condition $|g(k_1)| > |r(k_1)|$. Then, according to Rouche's theorem, the function $g(k_1) \pm r(k_1)$ has the same number of zeros as $g(k_1)$ inside C. It will be clear that $g(k_1)$, as a quasipolynomial of the second order, has two roots. Thus, the function $R(k_1)$ has four different roots.

Now some particular cases will be considered when loads are moving subcritically, transcritically and supercritically. Mixed cases can be obtained by the superposition of these "pure" cases.

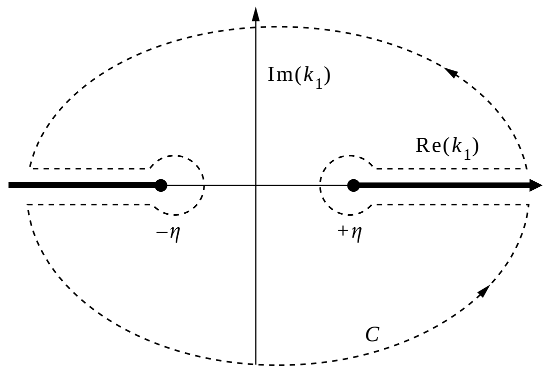

Figure 2. Branch points and cuts in the case $b > 0$, contour in k_1-plane.

3.1. SUBCRITICAL CASE ($\beta_j < \alpha$, 1)

If the velocity of the load is smaller than c_s, c_m ($\beta_j < \alpha$, 1), then equation (8) can be rewritten in the form

$$U_1^m(x, y, t) = -\frac{P_1}{2\pi}\int_{-\infty}^{\infty}$$

$$\times \frac{\exp\{\mathrm{i}(x+\beta_1 t)k_1\}}{\left(\left((1-\beta_1^2)k_1^2+\gamma\sqrt{(\alpha^2-\beta_1^2)k_1^2+1}\right)^2-(1-\beta_1^2)^2k_1^4\exp\left(-\frac{2d}{\alpha}\sqrt{(\alpha^2-\beta_1^2)k_1^2+1}\right)\right)}$$

$$\times\left\{\left((1-\beta_1^2)k_1^2+\gamma\sqrt{(\alpha^2-\beta_1^2)k_1^2+1}\right)\exp\left\{-\left|y-\frac{d}{2}\right|\frac{\sqrt{(\alpha^2-\beta_1^2)k_1^2+1}}{\alpha}\right\}\right.$$

$$\left.-(1-\beta_1^2)k_1^2\exp\left\{-\left|y+\frac{d}{2}\right|\frac{\sqrt{(\alpha^2-\beta_1^2)k_1^2+1}}{\alpha}\right\}\exp\left(-\frac{d}{\alpha}\sqrt{(\alpha^2-\beta_1^2)k_1^2+1}\right)\right\}\mathrm{d}k_1, \tag{12}$$

with $\mathrm{Re}\sqrt{(\alpha^2-\beta_1^2)k_1^2+1} > 0$. Equation (12) consists of a symmetrical part related to the load $\{y = d/2,\ x = -\beta_1 t\}$ and an asymmetrical part due to the presence of the other string. By using equation (12) one can easily obtain the expression for $U_2^m(x, y, t)$ by performing the substitutions

$$P_1 \to P_2, \qquad \beta_1 \to \beta_2, \qquad x + \beta_1 t \to x - \beta_2 t, \qquad |y \pm d/2| \to |y \mp d/2|. \tag{13}$$

The solution of the "two loads problem" is then given by equation (9).

In order to integrate equation (12) and the analogous equation for U_2^m, one has to determine the locations of the singular points of the integrand (12) in the complex k_1-plane (the analogous reasoning for U_2^m is omitted). The integrand of equation (12) is a multiple-valued function because of the presence of the radical $G(k_1) = \sqrt{(\alpha^2-\beta_1^2)k_1^2+1}$. Branch points occur when $G(k_1) = 0$. Thus, one has two branch points on the imaginary k_1-axis: $\eta_{1,2} = \pm\mathrm{i}/\sqrt{\alpha^2-\beta_1^2}$. Also the four zeros of the denominator of the integrand in equation (12) are located on the imaginary axis.

From a physical point of view the fact that the poles are on the imaginary axis corresponds to a moving load which does not radiate any elastic waves into the string. Further, it corresponds to the absence of wave processes in the membrane. Therefore, one can integrate equations (12) and (13) along the real k_1-axis using a standard numerical method. The results for two moments of time are shown in Figures 3(a, b). As confirmed in the figures, the moving loads excite a local eigenfield moving with the loads.

If alpha is reducing, $\alpha = (c_m/c_s) \rightarrow \beta$ and $(\beta < \alpha < 1)$, so that for $\beta = \alpha < 1$, one has a stationary eigenfield with singularities (with a jump in the first derivatives).

3.2. INTERACTION FORCE

Again two loads are considered moving subcritically in an opposite direction along the strings. Each load generates a spatially extended eigenfield that is moving with it. Therefore, the loads will affect each other through the membrane by means of the eigenfields, especially when the loads are getting close to one another, as depicted in Figure 4(a). This topic will be studied in this section.

A single load moving along one of the strings on the elastically supported membrane has an eigenfield which consists of a central–symmetrical part and an asymmetrical part in the frame of reference connected to the load, as shown earlier in equation (12). (Later on we will discuss the effect of the asymmetrical part.)

The central–symmetric part of the eigenfield is such that no additional force has to be introduced to maintain the constant motion of the load. Then the reaction of the elastic system N is vertical due to the symmetry and has no horizontal projection; see Figure 4(b).

An additional asymmetry of the eigenfield may occur, for instance, due to the presence of the other load or if the load excites elastic waves; see Figures 4(c, d). Due to the asymmetry in the superposed eigenfields a horizontal force $\mathbf{F}_f = -\mathbf{F}_r$ has to be applied to balance the reaction $\mathbf{F}_r$ (which is the horizontal projection of the string reaction $\mathbf{N}$ in the loading point) to maintain the defined law of constant motion of the load.

The load creates an elastic field of vertical displacements around itself, and then a certain force acts on every other object located in this field. The gradient of that elastic field is

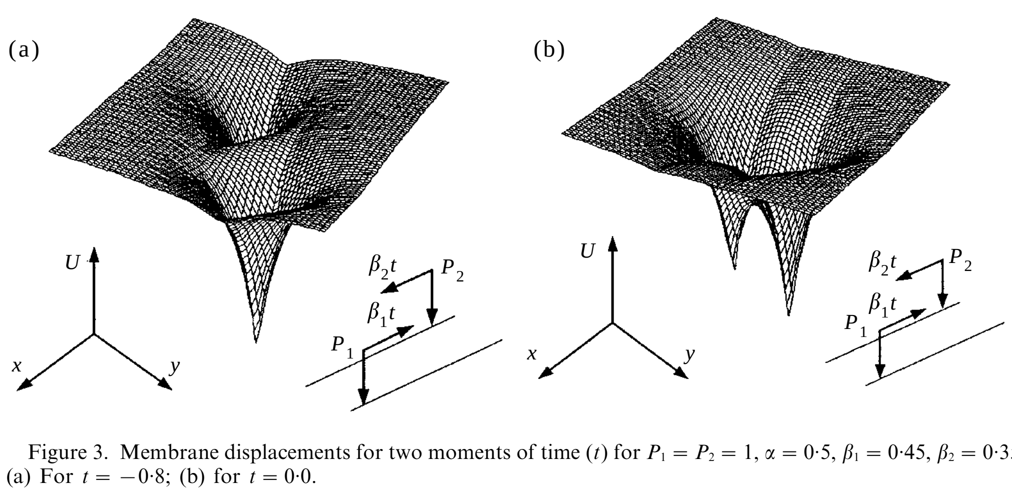

Figure 3. Membrane displacements for two moments of time (t) for $P_1 = P_2 = 1$, $\alpha = 0{\cdot}5$, $\beta_1 = 0{\cdot}45$, $\beta_2 = 0{\cdot}35$. (a) For $t = -0{\cdot}8$; (b) for $t = 0{\cdot}0$.

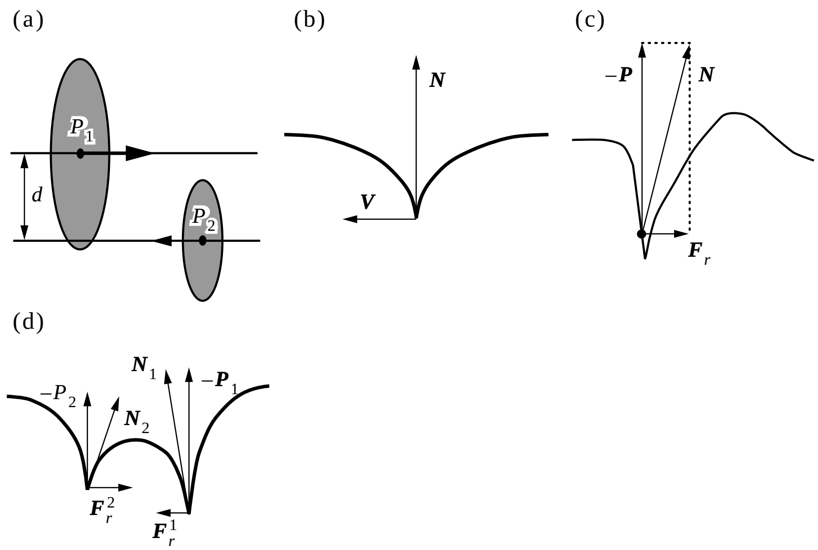

Figure 4. Qualitative pictures for the current topic. (a) Top view of two moving loads along strings on a membrane; the domains of the substantial deformations under the loads are shaded; (b) profile of the symmetric eigenfield in the "homogeneous" case; (c, d) profiles of the elastic system in cases when the load generates elastic waves (c), or two loads are placed near each other (d).

a measure of this force. This force can also be obtained by a geometrical analysis of Figures 4(c) or (d). Then the expressions for the interaction forces are given by [6]

$$\mathbf{F}_r^{21} = (F_{rx}^{21}, F_{ry}^{21}) = -P_1 \,\mathrm{grad}\, U_2^m (x = -\beta_1 t, y = d/2, t) = -P_1 \left(\frac{\partial}{\partial x} U_2^m, \frac{\partial}{\partial y} U_2^m \right) \Bigg|_{\substack{x = -\beta_1 t \\ y = d/2}},$$

$$\mathbf{F}_r^{12} = (F_{rx}^{12}, F_{ry}^{12}) = -P_2 \,\mathrm{grad}\, U_1^m (x = \beta_2 t, y = -d/2, t) = -P_2 \left(\frac{\partial}{\partial x} U_1^m, \frac{\partial}{\partial y} U_1^m \right) \Bigg|_{\substack{x = \beta_2 t \\ y = -d/2}}, \tag{14}$$

where $\mathbf{F}_r^{ij}$, $i, j = 1, 2$ is the interaction force acting at the jth load due the ith load. Note that the asymmetrical part of the eigenfield of the load, mentioned earlier, occurs due to the other string (we call it the internal asymmetry). It results in a constant repulsive lateral force with respect to the string. This force is negligibly small compared with the interaction force to be discussed now.

Now, the lateral component of the interaction force $F_{ry}^{ij}(t)$ with respect to the string will be studied.

For the case that the loads (P_1, P_2) are moving subcritically, $F_{ry}^{12}(t)$ is evaluated by using equations (14) and (12) for the displacements U_1^m for three distances between the strings and for three velocities of the first load. The results are shown in Figures 5(a, b).

As shown in the figures, the lateral component of the interaction force $F_{ry}^{12}(t)$ is positive. This implies that the strings will attract each other in the case when the two vertical loads have the same direction. The figures further show that the force increases when the distance between the loads decreases. In this sense the interaction force is analogous to Newton's gravitational interaction force [7]. Also an analogy can be made with respect to the law of Faraday from electromechanics; however, this analogy is more conceptual.

3.3. TRANSCRITICAL CASES ($\alpha < \beta_j < 1$ or $1 < \beta_j < \alpha$).

Case 1: $\alpha < \beta_1 < 1 (c_m < V_1, V_2 < c_s)$

Suppose that the first load is moving faster than the wave velocity in the membrane but slower than the wave velocity in the string ($\alpha < \beta_1 < 1$). Then equation (8) can be rewritten in the form

$$U_1^m(x, y, t) =$$

$$-\frac{P_1}{2\pi}\int_{-\infty}^{\infty}\frac{\exp\{\mathrm{i}(x+\beta_1 t)k_1\}}{\left(\left((1-\beta_1^2)k_1^2 - \mathrm{i}\gamma\sqrt{(\beta_1^2-\alpha^2)k_1^2-1}\right)^2 - (1-\beta_1^2)^2 k_1^4 \exp\left(\mathrm{i}\frac{2d}{\alpha}\sqrt{(\beta_1^2-\alpha^2)k_1^2-1}\right)\right)}$$

$$\times\left\{((1-\beta_1^2)k_1^2 - \mathrm{i}\gamma\sqrt{(\beta_1^2-\alpha^2)k_1^2-1})\exp\left\{\mathrm{i}\left|y-\frac{d}{2}\right|\frac{\sqrt{(\beta_1^2-\alpha^2)k_1^2-1}}{\alpha}\right\}\right.$$

$$\left.-(1-\beta_1^2)k_1^2\exp\left\{\mathrm{i}\left|y+\frac{d}{2}\right|\frac{\sqrt{(\beta_1^2-\alpha^2)k_1^2-1}}{\alpha}\right\}\exp\left(\mathrm{i}\frac{d}{\alpha}\sqrt{(\beta_1^2-\alpha^2)k_1^2-1}\right)\right\}\mathrm{d}k_1, \qquad (15)$$

with $\mathrm{Im}\sqrt{(\beta_1^2-\alpha^2)k_1^2-1} > 0$. The denominator of the integrand in equation (15) has four imaginary zeros in the complex k_1-domain, which have been calculated numerically for $\alpha = 0{\cdot}5$, $\beta = 0{\cdot}7$. This implies that the moving load does not excite elastic waves in the strings.

Two branch points $\eta_{1,2} = 1/\sqrt{\beta_1^2-\alpha^2}$ are located on the real k_1-axis. In order to evaluate the integral (15) by contour integration one has to investigate the locations of the branch points and zeros after introducing a small dissipation. One, therefore, introduces the member $2\delta U_t$ ($\delta \to +0$) in equation (2), which describes an additional viscous dissipation

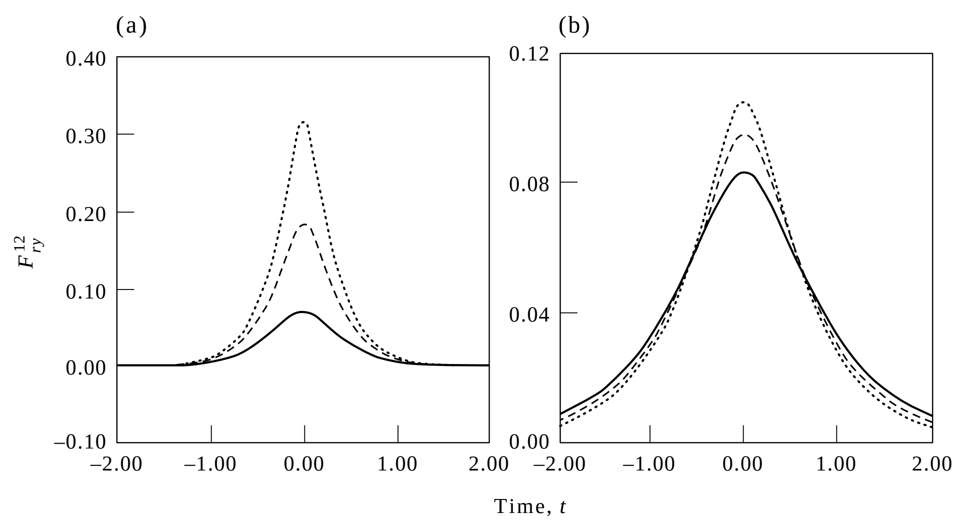

Figure 5. The y-component of the interaction force $F_r^{12}(t)$ for $\alpha = 0{\cdot}5$, $P_1 = P_2 = 1$. (a) For three distances d between the strings and $\beta_1 = \beta_2 = 0{\cdot}40$: . . . . , $d = 0{\cdot}4$; – – –, $d = 0{\cdot}6$; ——, $d = 1{\cdot}0$. (b) For three velocities β_1 of the first load and $\beta_2 = 0{\cdot}42$, $d = 1{\cdot}0$: . . . . , $\beta_1 = 0{\cdot}48$; – – –, $\beta_1 = 0{\cdot}40$; ——; $\beta_1 = 0{\cdot}20$. The time $t = 0{\cdot}0$ corresponds to the minimum distance between loads.

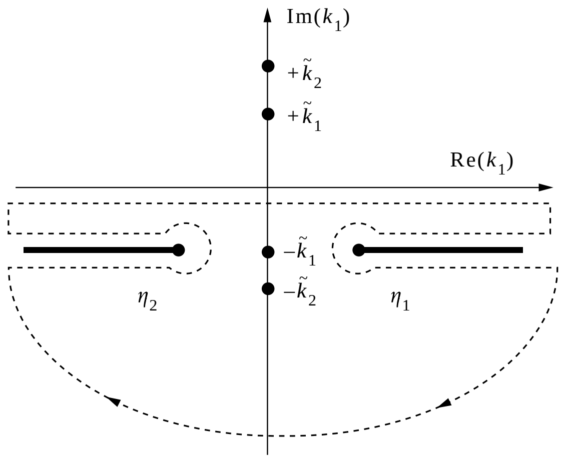

Figure 6. Branch points and cuts in the case $\alpha < \beta_1 < 1$; contour of integration in the lower half-plane.

in the foundation. After introducing this small dissipation the branch points move into the lower half-plane:

$$\eta_{1,2} = -\frac{i\delta\beta_1}{(\beta_1^2 - \alpha^2)} \pm \left(\frac{1}{\sqrt{\beta_1^2 - \alpha^2}} + O(\delta^2)\right), \qquad Q(k_1) = k_1^2 (\beta_1^2 - \alpha^2) + 2i\delta\beta_1 k_1 - 1,$$

$\mathrm{Im}\,(\sqrt{Q(k_1)}) > 0$ with $Q(\eta_{1,2}) = 0$. So it is appropriate to cut the plane along a line $\mathrm{Im}\,Q(k_1) = 0$ as qualitatively shown in Figure 6. Then the radical $\sqrt{Q(k_1)}$ has a positive imaginary part everywhere on the path of integration. Following Jordan's lemma [5]: for $\arg_1(\zeta_1, y) > 0$ and $\arg_2(\zeta_1, y) > 0$, one can close the path of integration (along the real axis) in the upper half-plane and for $\arg_1(\zeta_1, y)$, $\arg_2(\zeta_1, y) < 0$ in the lower half-plane, where

$$\arg_1(\zeta_1, y) = \left\{\zeta_1 + \left|y - \frac{d}{2}\right|\sqrt{\frac{\beta_1^2}{\alpha^2} - 1}\right\}, \qquad \arg_2(\zeta_1, y) = \left\{\zeta_1 + \left(d + \left|y + \frac{d}{2}\right|\right)\sqrt{\frac{\beta_1^2}{\alpha^2} - 1}\right\}, \tag{16}$$

and $\zeta_1 = x + \beta_1 t$. The expressions (16) describe the two Mach-cones in which the excited wave fields are confined.

Now one can reduce the integral (15) to a form suitable for numerical analysis; see the Appendix, equations (A2) and (A3). Figures 7(a, b) present the results of the numerical evaluation of equations (A2) and (A3). The figures show that the transcritically-moving load P_1 ($\alpha < \beta_1 < 1$) radiates elastic waves into the membrane. The wave field is located inside the cone analogous to the Mach-cone in acoustics, which satisfies the equation

$$\xi_1 = -\left|y - \frac{d}{2}\right|\sqrt{\frac{\beta_1^2}{\alpha^2} - 1} = -\left|y - \frac{d}{2}\right|\frac{1}{\tan(\theta)}, \quad \text{where } \sin(\theta) = \alpha/\beta_1 = c_m/V_1.$$

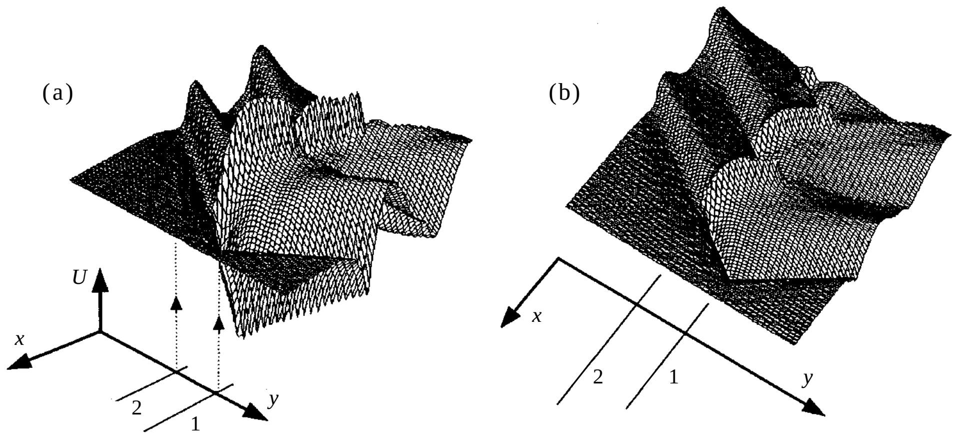

Figure 7. Different views of the membrane displacements for $\alpha = 0{\cdot}5$, $\beta_1 = 0{\cdot}7$, $d = 1{\cdot}0$. 1 and 2 denote the first and second strings.

The wave field reflecting with an opposite sign by the second string gives a secondary Mach-cone described by the equation

$$\xi_1 = -\left(d + \left|y + \frac{d}{2}\right|\right)\sqrt{\frac{\beta_1^2}{\alpha^2} - 1} = -\left(d + \left|y + \frac{d}{2}\right|\right)\frac{1}{\tan(\theta)},$$

see also Figure 8. A part of the wave energy is transferred to the outer membrane. Furthermore, the secondary Mach-cone reflected partly by the first string again forms a third Mach-cone and so on. The amplitudes of successive Mach-cones decrease with the number of reflections due to the transfer of energy to the membrane. As can be seen in Figure 7(a), the second string reflects the wave field effectively since no wave energy propagates along the string. Nevertheless, the waves in the membrane excite a vibration

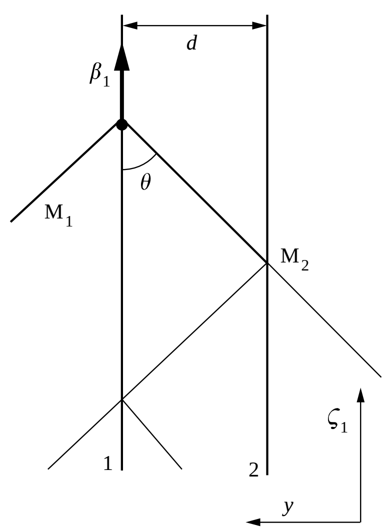

Figure 8. The moving load (β_1) generates a wave field which is confined inside the Mach-cone M_1, the waves are partly reflected by the second string 2, and form the secondary Mach-cone M_2, and so on.

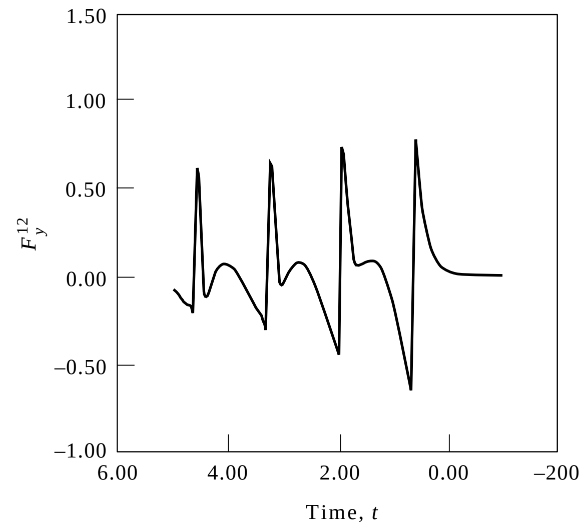

Figure 9. The lateral force which is acting at the second load when it is moving in the field of the transcritically moving first load, for the parameters $\alpha = 0{\cdot}5$, $\beta_1 = 0{\cdot}7$, $\beta_2 = 0{\cdot}3$, $d = 1{\cdot}0$.

in the string and the amplitude of this vibration attenuates with increasing distance from the source point.

Now, the lateral force is determined which is acting at the second load when it is moving subcritically in the given elastic field generated by the transcritically moving first load. According to equation (14), using equations (A2) and (A3) and a standard numerical routine, the numerical expression for the lateral force was derived. The results of the evaluations are shown in Figure 9. The graph shows that the lateral force that is acting at the second load has an impact character. The first quasi-impact is the most powerful one because the load is crossing the border of the first Mach-cone. The next quasi-impacts are smaller and smoother than the previous ones and they originate due to the crossing of the borders of the succeeding Mach-cones.

From the practical point of view, such behaviour of the lateral force can be quite dangerous for the stability of the passing trains. However, this model cannot give any qualitative estimation which is suitable for practice.

Case 2: $1 < \beta_1 < \alpha (c_s < V_1,\ V_2 < c_m)$

In this case the displacement of the system accounting for the viscous dissipation in the foundation is given by

$$U^m_{1,\delta}\,(\zeta_1 = x + \beta_1\, t,\, y)$$

$$= \frac{P_1}{2\pi}\int_{-\infty}^{\infty} \frac{\exp\{\mathrm{i}\zeta_1\, k_1\}}{\left(((\beta_1^2-1)k_1^2-\gamma\sqrt{\sigma(k_1,\delta)})^2-(\beta_1^2-1)^2k_1^4\exp\left(-\dfrac{2d}{\alpha}\sqrt{\sigma(k_1,\delta)}\right)\right)}$$

$$\times\left\{((\beta_1^2-1)k_1^2-\gamma\sqrt{\sigma(k_1,\delta)})\exp\left\{-\left|y-\frac{d}{2}\right|\frac{\sqrt{\sigma(k_1,\delta)}}{\alpha}\right\}\right.$$

$$-(\beta_1^2-1)k_1^2\exp\left\{-\left(d+\left|y+\frac{d}{2}\right|\right)\frac{\sqrt{\sigma(k_1,\delta)}}{\alpha}\right\}\right\}\,\mathrm{d}k_1, \tag{17}$$

with $\mathrm{Re}\sqrt{\sigma(k_1,\delta)}>0$, where $\sigma(k_1,\delta)=k_1^2(\alpha^2-\beta_1^2)-2\mathrm{i}\delta k_1+1$, $U_1^m(\zeta_1,y)=\lim_{\delta\to+0}U_{1,\delta}^m(\zeta_1=x+\beta_1 t,y)$. The integrand of equation (17) has two branch cuts, which are located in the upper and lower complex half-plane k_1,

$$\eta_{1,2}=\mathrm{i}\,\frac{\delta\beta_1}{(\alpha^2-\beta_1^2)}\pm\mathrm{i}\left(\frac{1}{\sqrt{\alpha^2-\beta_1^2}}+O(\delta^2)\right),$$

and four poles in the lower half-plane, following from analysis of the denominator of equation (18), as depicted in Figure 10. If one let $\delta\to+0$ the poles move to the real k_1-axis and are related to the wave processes in the strings.

After these preliminary remarks one may integrate expression (17) by contour integration. Noting that for $\zeta_1>0$ the path of integration can be closed in the upper half-plane and for $\zeta_1<0$ in the lower complex k_1-half-plane, one can find the displacements of the system; see equations (A4) and (A5) in the Appendix.

Figures 11(a, b) represent the results of the numerical calculations of expressions (A4) and (A5) for two distances between the strings. The figures show that the transcritically-moving load P_1 $(1<\beta_1<\alpha)$ generates waves in the first string. Further, the oscillating string excites waves into the membrane. These waves are attenuated during propagation through the membrane, and are exciting undamped waves into the second string (due to the system dispersion properties). Therefore, the amplitude of that "secondary wave" depends on the distance between the strings; see Figure 11(b). It can further be shown that the strings are interacting through the membrane with a characteristic retardation in time due to the finite wave velocity in the membrane.

In order to describe qualitatively the interaction between the strings, one can introduce a parameter $\lambda\cong 1/\sqrt{1-(\beta_1^2/\alpha^2)}$, which can be obtained from equation (17). Figures 12 (a, b) represent the displacements of the strings for $d=1{\cdot}0$ and $d=3{\cdot}0$. They can be interpreted as the motion of two coupled oscillators. When the distance is substantially smaller than $\lambda\approx 3{\cdot}3$, then one has a strong interaction between the strings; see

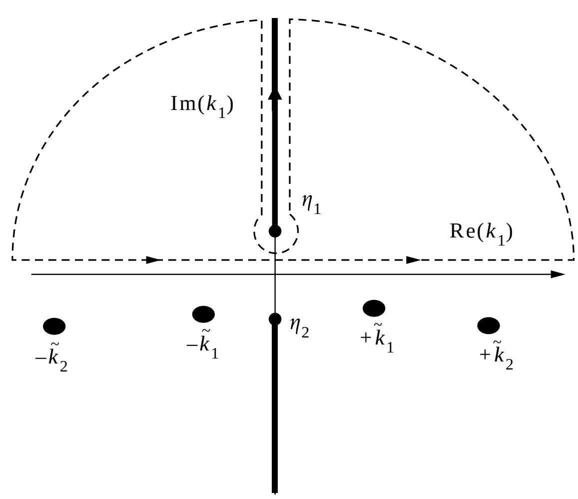

Figure 10. Branch points ($\delta\to+0$) and cuts in the case $1<\beta_1<\alpha$; contour of integration in the upper half-plane.

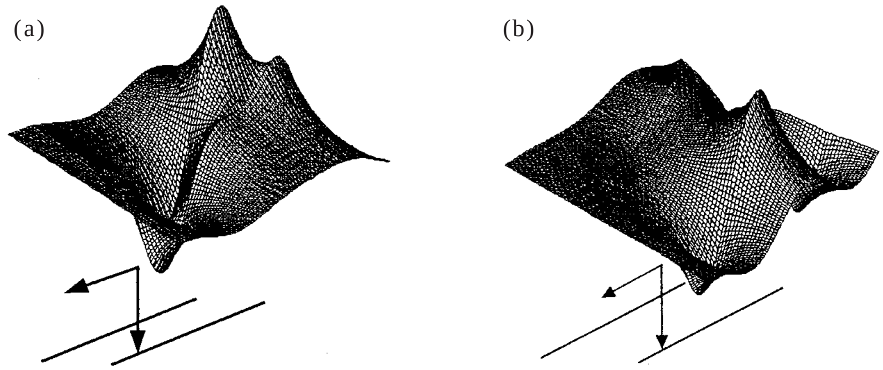

Figure 11. Displacement of the membrane for two distances between the strings for $\alpha = 1{\cdot}8$, $\beta_1 = 1{\cdot}5$. (a) For $d = 1{\cdot}0$; (b) for $f = 3{\cdot}0$.

Figure 12(a). Figure 12(b) shows the case where the distances are approximately equal to λ.

For the present case, the lateral force which acts at a subcritically moving load in the elastic field exerted by a transcritically moving other load will be investigated. Figure 13 presents the results of numerical calculations by using equations (14), (A4) and (A5) for the distances $d = 1{\cdot}0$, $3{\cdot}0$ between the strings. As shown in the figures, the lateral force oscillates with respect to time and the amplitude of the force obviously depends on the parameter d.

3.4. SUPERCRITICAL CASE $(1, \alpha < \beta_j)$

Now the velocity of the load P_1 is higher than the wave velocity in the membrane and in the strings. In this case, the moving load generates elastic waves both in the membrane

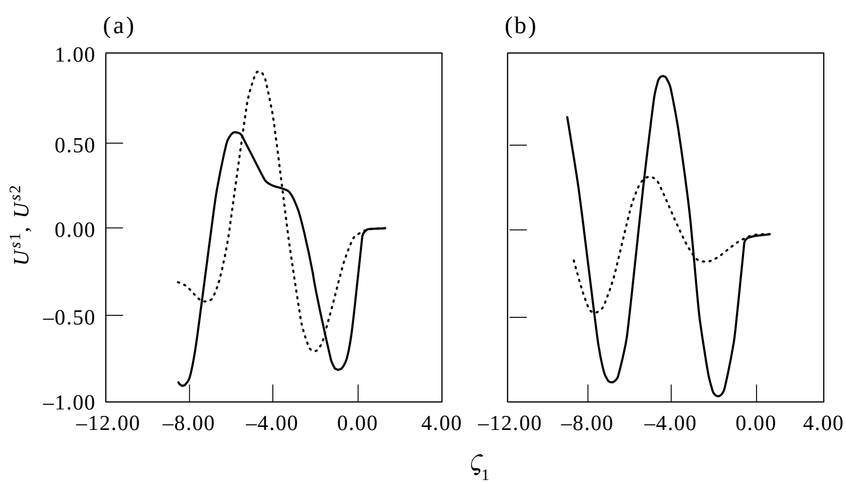

Figure 12. Displacement of the strings for two distances between the strings for $\alpha = 1{\cdot}8$, $\beta_1 = 1{\cdot}5$, $\lambda \approx 3{\cdot}3$. (a) For $d = 1{\cdot}0$; (b) for $d = 3{\cdot}0$: ——, U^{s1}; $\cdots\cdot$, U^{s2}.

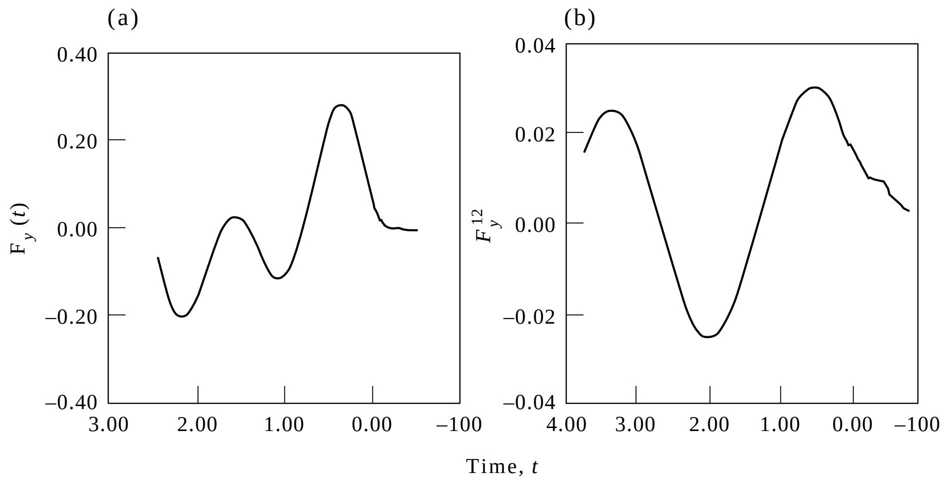

Figure 13. The lateral force which is acting at the second load when it is subcritically moving in the field of a transcritically moving first load, for $\alpha = 1{\cdot}8$, $\beta_1 = 1{\cdot}5$, $\beta_2 = 0{\cdot}8$. (a) For $d = 1{\cdot}0$; (b) for $d = 3{\cdot}0$.

and in the strings. The displacements of the system are described by the expression

$$U^m_{1,\delta}(\zeta_1 = x + \beta_1 t, y)$$

$$= \frac{P_1}{2\pi}\int_{-\infty}^{\infty} \frac{\exp\{i\zeta_1 k_1\}}{\left(((\beta_1^2 - 1)k_1^2 + i\gamma\sqrt{\sigma(k_1,\delta)})^2 - (\beta_1^2-1)^2 k_1^4 \exp\left(i\,\frac{2d}{\alpha}\sqrt{\sigma(k_1,\delta)}\right)\right)}$$

$$\times\left\{((\beta_1^2 - 1)k_1^2 + i\gamma\sqrt{\sigma(k_1,\delta)})\exp\left\{i\left|y - \frac{d}{2}\right|\frac{\sqrt{\sigma(k_1,\delta)}}{\alpha}\right\}\right.$$

$$\left. -(\beta_1^2 - 1)k_1^2 \exp\left\{i\left(d + \left|y + \frac{d}{2}\right|\right)\frac{\sqrt{\sigma(k_1,\delta)}}{\alpha}\right\}\right\}\mathrm{d}k_1, \quad \text{with } \operatorname{Im}\sqrt{\sigma(k_1,\delta)} > 0, \tag{18}$$

where $\sigma(k_1, \delta) = k_1^2(\beta_1^2 - \alpha^2) + 2i\delta k_1 - 1$, $U_1^m(\zeta_1, y) = \lim_{\delta\to+0} U^m_{1,\delta}(\zeta_1 = x + \beta_1 t, y)$. The integrand of expression (18) has two branch points $\eta_{1,2}$ and four poles $\pm\tilde{k}_{1,2}$ in the lower k_1-half-plane; see Figure 14. The branch points are $\eta_{1,2} = -i\delta\beta_1/\vartheta^2 \pm (1/\vartheta - O(\delta^2))$, $\vartheta = \sqrt{\beta_1^2 - \alpha^2}$. Furthermore, two of the poles, say $\pm\tilde{k}_1$, coincide with the branch points but after introducing an additional dissipation in the strings one can separate them. The branch cuts have been chosen along the lines $\operatorname{Im}\sqrt{\sigma(k_1, \delta\to 0)} = 0$, as shown in Figure 14. The present location of the poles and branch points corresponds to wave processes in the strings and the membrane inside the Mach-cones, $\arg_1(\zeta_1, y)$, $\arg_2(\zeta_1, y) < 0$, where

$$\arg_1(\zeta_1, y) = \left\{\zeta_1 + \left|y + \frac{d}{2}\right|\sqrt{\frac{\beta_1^2}{\alpha^2} - 1}\right\}, \qquad \arg_2(\zeta_1, y) = \left\{\zeta_1 + \left(d + \left|y + \frac{d}{2}\right|\right)\sqrt{\frac{\beta_1^2}{\alpha^2} - 1}\right\}.$$

Outside these cones, the system of strings and membrane is not excited. After these considerations, one can reduce the integral (18) to a form which is suitable for numerical

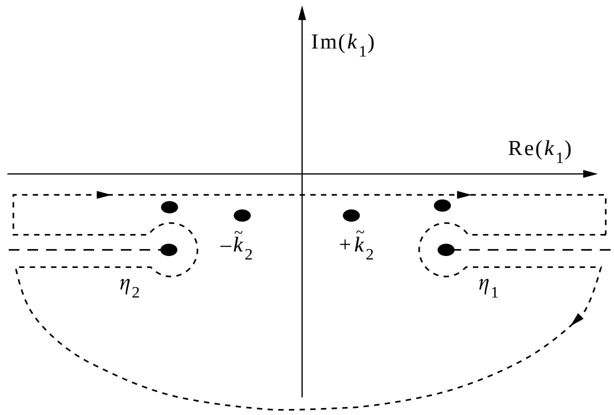

Figure 14. Branch points and cuts in the supercritical case ($\beta_j > \alpha$, 1); contour of integration in the lower half-plane.

calculations; see equations (A6) and (A7) in the Appendix. The results of the numerical calculations are shown in Figures 15(a, b). As can be seen from the graphs, the wave field is located inside the Mach-cones ($\arg_{1,2}(\zeta_1, y) < 0$). The waves, reflecting several times between strings, are attenuating due to the transfer of part of their energy into the membrane outside the strings. Furthermore, for the "two-string system" the wave field does not exhibit a jump in the displacements on the Mach-cone border, as was found for the "one-string system" [4].

Now, the lateral force is determined, which is acting at the second subcritically moving load in the elastic field generated by a supercritically moving first load. The graph of this force is presented in Figure 16. As shown in the figure, the lateral force has an impact character. Before passing the border of the Mach-cone ($t < t^*$, $t^* > 0$) the force is zero because the system is not yet excited there. On the cone border the lateral force has a jump due to the discontinuity of the first derivative of the membrane displacement. In other words, when one of the loads is crossing the border of the Mach-cone, where displacements are varying very fast, the load "feels" impact.

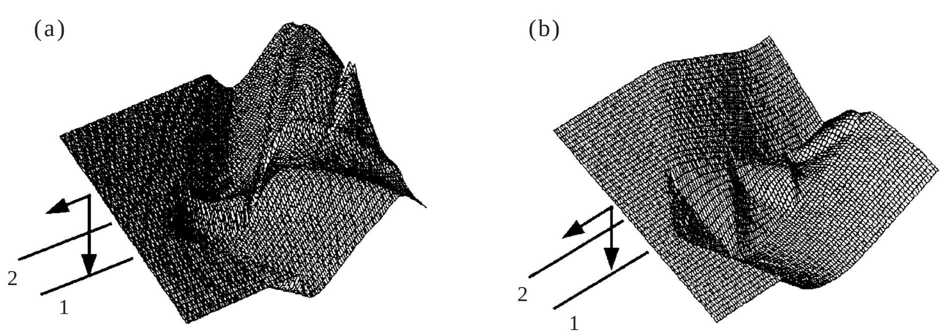

Figure 15. The displacements of the system when the load is supercritically moving along the first string 1, $d = 1{\cdot}0$. (a) For $\alpha = 0{\cdot}9$, $\beta_1 = 1{\cdot}5$; (b) for $\alpha = 1{\cdot}5$, $\beta_1 = 1{\cdot}9$.

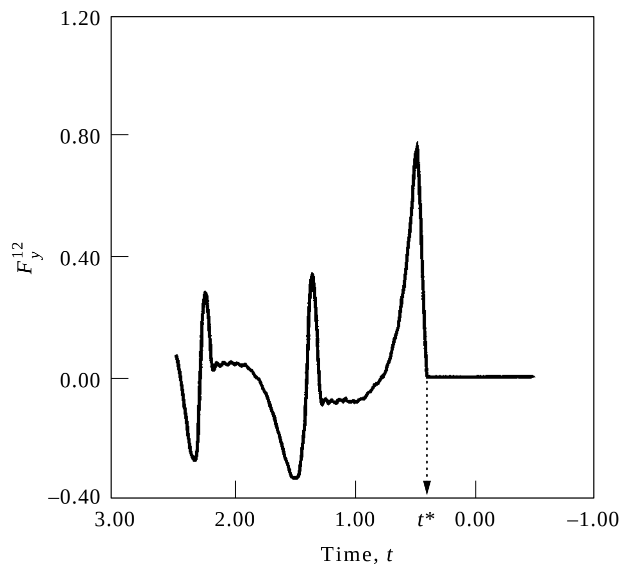

Figure 16. The lateral force which is acting at the second subcritically moving load in the field of a supercritically moving first load, for $\alpha = 0{\cdot}9$, $d = 1{\cdot}0$, $\beta_1 = 1{\cdot}5$, $\beta_2 = 0{\cdot}5$.

4. CONCLUSIONS

In this paper the uniform motion of two constant loads moving in opposite directions along two parallel strings on an elastically supported membrane has been studied. The steady state behaviour of the system has been derived. For subcritical, transcritical and supercritical load velocities the displacement fields have been calculated and presented as graphs.

The results show that the presence of a second string essentially affects the dynamic behaviour of the system (compared to that of the "one-string case" [4]), especially when the dimensionless distance between the strings is relatively small.

A subcritically moving load along a string is shown to have a stationary eigenfield of vertical displacements with a symmetrical and an asymmetrical part with respect to the frame of reference related to the load. By virtue of this last part, repulsive lateral force (with respect to the string) acts at the load.

In the case of transcritical and supercritical motion, the load generates waves in the system. The resulting vertical displacements in the system show a quite inhomogeneous character with large spatial gradients.

Further, the lateral components of the interaction force has been determined, which is acting at the loads, when two loads are moving in opposite directions along the strings. The character and amplitude of this interaction force depends on the velocity of the loads and the distance between the strings respectively. In particular, cases have been studied when one of the loads is moving subcritically in a field of the other load. For subcritically moving loads, the lateral force depends smoothly on time and it corresponds to an attraction between the loads and strings. Hence, it has a qualitative analogy with gravitational interaction.

When one of the loads is moving supercritically or transcritically then the direction of the lateral force that is acting at the other load is fluctuating in time. Moreover both in one of the transcritical cases, in which the wave velocity in the string is higher than the membrane wave velocity ($\alpha < \beta_j < 1$), and in the supercritical case this force is shown to have an impulsive character.

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APPENDIX

A.1. GENERAL SOLUTION

$$U_2^m(x, y, t) = \frac{P_2}{2\pi}\int_{-\infty}^{\infty}$$

$$\times \frac{\exp\{\mathrm{i}(x-\beta_2 t)k_1\}}{\left(((\beta_2^2-1)k_1^2+\mathrm{i}\gamma\sqrt{(\beta_2^2-\alpha^2)k_1^2-1})^2-(\beta_2^2-1)^2k_1^4\exp\left(\mathrm{i}\,\frac{2d}{\alpha}\sqrt{(\beta_2^2-\alpha^2)k_1^2-1}\right)\right)}$$

$$\times\left\{((\beta_2^2-1)k_1^2+\mathrm{i}\gamma\sqrt{(\beta_2^2-\alpha^2)k_1^2-1})\exp\left\{\mathrm{i}\left|y+\frac{d}{2}\right|\frac{\sqrt{(\beta_2^2-\alpha^2)k_1^2-1}}{\alpha}\right\}\right.$$

$$\left.-(\beta_2^2-1)k_1^2\exp\left\{\mathrm{i}\left|y-\frac{d}{2}\right|\frac{\sqrt{(\beta_2^2-\alpha^2)k_1^2-1}}{\alpha}\right\}\exp\left(\mathrm{i}\,\frac{d}{\alpha}\sqrt{(\beta_2^2-\alpha^2)k_1^2-1)}\right)\right\}\mathrm{d}k_1. \quad \text{(A1)}$$

with $\mathrm{Im}\sqrt{(\beta_2^2-\alpha^2)k_1^2-1} > 0$.

A.2. TRANSCRITICAL CASE 1

For $\arg_1(\zeta_1, y)$, $\arg_2(\zeta_1, y) > 0$,

$$U_1^m(\zeta_1, y) = -P_1 \sum_{+\tilde{k}_1, +\tilde{k}_2} \mathrm{i} \frac{(k_1^2\chi^2 - \mathrm{i}\gamma\sqrt{k_1^2\vartheta - 1})}{\dfrac{\mathrm{d}}{\mathrm{d}k_1} B_+(k_1)} \exp\left\{\mathrm{i}\left(k_1\zeta + \left|y - \frac{d}{2}\right| \frac{\sqrt{k_1^2\vartheta^2 - 1}}{\alpha}\right)\right\}$$

$$+ P_1 \sum_{+\tilde{k}_1, +\tilde{k}_2} \mathrm{i} \frac{(k_1^2\chi^2)}{\dfrac{\mathrm{d}}{\mathrm{d}k_1} B_+(k_1)} \exp\left\{\mathrm{i}\left(k_1\zeta + \left(d + \left|y - \frac{d}{2}\right|\right) \frac{\sqrt{k_1^2\vartheta^2 - 1}}{\alpha}\right)\right\}. \tag{A2}$$

For $\arg_1(\zeta_1, y)$, $\arg_2(\zeta_1, y) < 0$,

$$U_1^m(\zeta_1, y) = P_1 \sum_{-\tilde{k}_1, -\tilde{k}_2} \mathrm{i} \frac{(k_1^2\chi^2 - \mathrm{i}\gamma\sqrt{k_1^2\vartheta^2 - 1})}{\dfrac{\mathrm{d}}{\mathrm{d}k_1} B_+(k_1)} \exp\left\{\mathrm{i}\left(k_1\zeta + \left|y - \frac{d}{2}\right| \frac{\sqrt{k_1^2\vartheta^2 - 1}}{\alpha}\right)\right\}$$

$$- P_1 \sum_{-\tilde{k}_1, \tilde{k}_2} \mathrm{i} \frac{(k_1^2\chi^2)}{\dfrac{\mathrm{d}}{\mathrm{d}k_1} B_+(k_1)} \exp\left\{\mathrm{i}\left(k_1\zeta + \left(d + \left|y - \frac{d}{2}\right|\right) \frac{\sqrt{k_1^2\vartheta^2 - 1}}{\alpha}\right)\right\}$$

$$- I_{Br}(\zeta_1, y), \tag{A3}$$

where $\chi^2 = 1 - \beta_1^2$, $\vartheta^2 = \beta_1^2 - \alpha^2$,

$$I_{Br}(\zeta_1, y) = A \int_\eta^\infty \mathrm{Re}\left\{\mathrm{i}\sin(\kappa\zeta_1)\left(\frac{(\kappa\chi)^2 + \mathrm{i}\gamma\sqrt{(\kappa\vartheta)^2 - 1}}{B_-(\kappa)} \exp(-f_1(y, \kappa))\right.\right.$$

$$\left.\left. - \frac{(\kappa\chi)^2 - \mathrm{i}\gamma\sqrt{(\kappa\vartheta)^2 - 1}}{B_+(\kappa)} \exp(f_1(y, \kappa))\right)\right\} \mathrm{d}\kappa + A \int_\eta^\infty \mathrm{Re}\left\{\mathrm{i}\sin(\kappa\zeta_1)\right.$$

$$\left. \times \left(\frac{(\kappa\chi)^2}{B_-(\kappa)} \exp(-f_2(y, \kappa)) - \frac{(\kappa\chi)^2}{B_+(\kappa)} \exp(f_2(y, \kappa))\right)\right\} \mathrm{d}\kappa,$$

with $A = P_1/\pi$, $\eta = 1/\sqrt{\beta_1^2 - \alpha^2}$,

$$f_1(y, \kappa) = \mathrm{i}\left|y - \frac{d}{2}\right| \frac{\sqrt{(\kappa\vartheta)^2 - 1}}{\alpha}, \qquad f_2(y, \kappa) = \mathrm{i}\left(d + \left|y - \frac{d}{2}\right|\right) \frac{\sqrt{(\kappa\vartheta)^2 - 1}}{\alpha},$$

$$B_\pm(\kappa = k_1) = (k_1^2\chi^2 \mp \mathrm{i}\gamma\sqrt{k_1^2\vartheta^2 - 1})^2 - (k_1\chi)^4 \exp\left\{\pm\mathrm{i}\left(\frac{2d}{\alpha}\right)\sqrt{k_1^2\vartheta^2 - 1}\right\}.$$

A.3. TRANSCRITICAL CASE 2

For $\zeta_1 > 0$,

$$U_1^m(\zeta_1 = x + \beta_1 t, y) = A \int_\eta^\infty \mathrm{Re}\left\{ \mathrm{i}\exp(-\kappa\zeta_1)\left(\frac{(\kappa\chi)^2 + \mathrm{i}\gamma\sqrt{(\kappa\vartheta)^2 - 1}}{B_+(\kappa)} \exp(f_1(y,\kappa)) \right.\right.$$

$$\left.\left. - \frac{(\kappa\chi)^2 - \mathrm{i}\gamma\sqrt{(\kappa\vartheta)^2 - 1}}{B_-(\kappa)} \exp(f_1(y,\kappa)) \right) \right\} \mathrm{d}\kappa + A \int_\eta^\infty \mathrm{Re}\,\{\mathrm{i}\exp(-\kappa\zeta_1)$$

$$\times \left. \left(\frac{(\kappa\chi)^2}{B_+(\kappa)} \exp(f_2(y,\kappa)) - \frac{(\kappa\chi)^2}{B_-(\kappa)} \exp(f_2(y,\kappa)) \right) \right\} \mathrm{d}\kappa. \tag{A4}$$

For $\zeta_1 < 0$,

$$U_1^m(\zeta_1, y) = -P_1 \sum_{\pm\bar{k}_1, \pm\bar{k}_2} \mathrm{i} \frac{(k_1^2\chi^2 - \gamma\sqrt{k_1^2\vartheta^2 + 1})}{\frac{\mathrm{d}}{\mathrm{d}k_1} B(k_1)} \exp\left\{ \mathrm{i}k_1\zeta - \left| y - \frac{d}{2} \right| \frac{\sqrt{k_1^2\vartheta^2 + 1}}{\alpha} \right\}$$

$$+ P_1 \sum_{\pm\bar{k}_1, \pm\bar{k}_2} \mathrm{i} \frac{(k_1^2\chi^2)}{\frac{\mathrm{d}}{\mathrm{d}k_1} B(k_1)} \exp\left\{ \mathrm{i}k_1\zeta - \left(d + \left| y - \frac{d}{2} \right| \right) \frac{\sqrt{k_1^2\vartheta^2 - 1}}{\alpha} \right\} - I_{Br}, \tag{A5}$$

where

$$A = P_1/(2\pi), \quad \chi = \sqrt{\beta_1^2 - 1}, \quad \vartheta = \sqrt{\alpha^2 - \beta_1^2},$$

$$f_1(y,\kappa) = \left| y - \frac{d}{2} \right| \frac{\sqrt{(\kappa\vartheta)^2 + 1}}{\alpha}, \qquad f_2(y,\kappa) = \left(d + \left| y - \frac{d}{2} \right| \right) \frac{\sqrt{(\kappa\vartheta)^2 + 1}}{\alpha},$$

and

$$B(k_1) = ((k_1\chi)^2 - \gamma\sqrt{(k_1\vartheta)^2 + 1}) - (k_1\chi)^4 \exp\left(-\frac{2d}{\alpha}\sqrt{(k_1\vartheta)^2 + 1} \right),$$

$$B_+(\kappa) = ((\kappa\chi)^2 + \mathrm{i}\gamma\sqrt{(\kappa\vartheta)^2 - 1})^2 - (\kappa\chi)^4 \exp\left(\mathrm{i}\frac{2d}{\alpha}\sqrt{(\kappa\vartheta)^2 - 1} \right),$$

$$B_-(\kappa) = ((\kappa\chi)^2 - \mathrm{i}\gamma\sqrt{(\kappa\vartheta)^2 - 1})^2 - (\kappa\chi)^4 \exp\left(-\mathrm{i}\frac{2d}{\alpha}\sqrt{(\kappa\vartheta)^2 - 1} \right).$$

I_{Br} is the integral along the branch cut, which has the same form as the two integrals in the expression for $\zeta_1 > 0$ but one has to substitute $\zeta_1 \to -\zeta_1$.

A.4. SUPERCRITICAL CASE

For $\arg_1(\zeta_1, y),\ \arg_2(\zeta_1, y) > 0$,

$$U^m_{1,\delta}(\zeta_1 = x + B_1 t, y) = 0. \tag{A6}$$

For $\arg_1(\zeta_1, y),\ \arg_2(\zeta_1, y) < 0$,

$$\begin{aligned}
U^m_1(\zeta_1, y) &= P_1 \sum_{\pm\bar{k}_1, \pm\bar{k}_2} \mathrm{i}\, \frac{(k_1^2 \chi^2 + \mathrm{i}\gamma\sqrt{k_1^2 \vartheta^2 - 1})}{\dfrac{\mathrm{d}}{\mathrm{d}k_1} B_+(k_1)} \exp\left\{\mathrm{i}\left(k_1 \zeta + \left|y - \frac{d}{2}\right| \frac{\sqrt{k_1^2 \vartheta^2 - 1}}{\alpha}\right)\right\} \\
&\quad - P_1 \sum_{\pm\bar{k}_1, \pm\bar{k}_2} \mathrm{i}\, \frac{(k_1^2 \chi^2)}{\dfrac{\mathrm{d}}{\mathrm{d}k_1} B_+(k_1)} \exp\left\{\mathrm{i}\left(k_1 \zeta + \left(d + \left|y - \frac{d}{2}\right|\right) \frac{\sqrt{k_1^2 \vartheta^2 - 1}}{\alpha}\right)\right\} \\
&\quad + A \int_\eta^\infty \mathrm{Re}\, \left\{\mathrm{i} \sin(\kappa\zeta_1)\left(\frac{(\kappa\chi)^2 + \mathrm{i}\gamma\sqrt{(\kappa\vartheta)^2 - 1}}{B_+(\kappa)} \exp(\mathrm{i}f_1(y, \kappa)) \right.\right. \\
&\quad \left.\left. - \frac{(\kappa\chi)^2 - \mathrm{i}\gamma\sqrt{(\kappa\vartheta)^2 - 1}}{B_-(\kappa)} \exp(-\mathrm{i}f_1(y, \kappa))\right)\right\} \mathrm{d}\kappa \\
&\quad + A \int_\eta^\infty \mathrm{Re}\, \left\{\mathrm{i} \sin(\kappa\zeta_1)\left(\frac{(\kappa\chi)^2}{B_-(\kappa)} \exp(\mathrm{i}f_2(y, \kappa)) \right.\right. \\
&\quad \left.\left. - \frac{(\kappa\chi)^2}{B_+(\kappa)} \exp(-\mathrm{i}f_2(y, \kappa))\right)\right\} \mathrm{d}\kappa,
\end{aligned} \tag{A7}$$

where

$$A = P_1/(2\pi), \qquad \eta = 1/\sqrt{\beta_1^2 - \alpha^2}, \qquad \chi = \sqrt{\beta_1^2 - 1}, \qquad \vartheta = \sqrt{\beta_1^2 - \alpha^2},$$

$$f_1(y, \kappa) = \left|y - \frac{d}{2}\right| \frac{\sqrt{(\kappa\vartheta)^2 - 1}}{\alpha}, \qquad f_2(y, \kappa) = \left(d + \left|y - \frac{d}{2}\right|\right) \frac{\sqrt{(\kappa\vartheta)^2 - 1}}{\alpha},$$

$$B_+(k_1) = ((k_1 \chi)^2 + \mathrm{i}\gamma\sqrt{(k_1 \vartheta)^2 - 1})^2 - (k_1 \chi)^4 \exp\left(\mathrm{i}\, \frac{2d}{\alpha} \sqrt{(k_1 \vartheta)^2 - 1}\right),$$

$$B_-(k_1) = ((k_1 \chi)^2 - \mathrm{i}\gamma\sqrt{(k_1 \vartheta)^2 - 1})^2 - (k_1 \chi)^4 \exp\left(-\mathrm{i}\, \frac{2d}{\alpha} \sqrt{(k_1 \vartheta)^2 - 1}\right).$$